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RETURN OF THE ICECREAM MEN. A DISCRETE HOTELLING GAME

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Biographical Notes

Nabi Abudaldah graduated in environmental economics at the Wageningen University with special interests in modelling, simulation and optimization. During his studies he participated in the MIT Energy Initiative as an author on the Future of the Electric Grid Study. He concluded his Master's degree in economics on the subject of the spread of energy efficient technologies using agent-based modelling methodologies. After graduation, he has been working as a technology consultant at various energy suppliers and trading companies in the Dutch energy markets. His current occupation is at Accenture focusing on energy markets and simulation technologies.

Wim Heijman (1953) received MSc degrees respectively in Economics and Human Geography from Tilburg University and the University of Utrecht in the Netherlands. He received his PhD degree from Wageningen University. In 2000 he was appointed Professor of Regional Economics at the latter university. His major research topics are rural development, landscape economics, and the assessment of the regional economic impact of tourism. Prof. Heijman was awarded an honorary doctor's degree by the University of Debrecen in 1999. In 2004, 2014 and 2015 he received honorary professor titles from the Czech University of Life Sciences in Prague, the Mongolian University of Life Sciences in Ulaanbaatar and the Stavropol Agrarian University respectively.

Pieter W. Heringa holds a bachelor's and master's degree (both *cum laude*) in Rural Development Economics from Wageningen University. He worked as junior researcher at the Rathenau Instituut for the Science System Assessment department; this work has resulted in a PhD thesis about patterns of collaborative knowledge production at Delft University of Technology. Pieter is currently employed as senior policy officer for the Dutch Ministry of Economic Affairs, at the Knowledge and Innovation department, where he is mainly involved in the governance of organisations for applied research.

Pierre v. Mouche (1959) studied theoretical physics, mathematics and astronomy at the University of Nijmegen. He received his PhD in mathematics from the University of Utrecht under the supervision of Hans Duistermaat. Since 1989 he holds a position in

economics at Wageningen University. His scientific interest includes mathematical economics and in particular the structure of the equilibrium set of aggregative games.

Abstract

We consider a finite symmetric game in strategic form between two players which can be interpreted as a discrete variant of the Hotelling game in a one or two-dimensional space. As the analytical investigation of this game is tedious, we simulate with Maple and formulate some conjectures. In addition we present a short literature overview.

Keywords: discrete Hotelling game, Maple, Nash equilibrium

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1. Introduction

For over eighty years, many people have been intrigued by one single article by professor Harold Hotelling. The most fascinating thing about Hotelling's model is perhaps its simple appearance at first sight, but despite this apparent simplicity, it touches upon so many issues, that new variations of Hotelling's model continue to be published until today. However, the elegance and simple nature of Hotelling's model also limit its value for real world applications. Real-world location questions often involve two dimensional spaces, multiple actors, a heterogeneous demand over space, and (policy) restrictions on sites where firms are able to be located. As such it is important to extend the models with more sophisticated specifications, as earlier work has shown that even minor changes in the assumptions of the model sometimes leads to enormous changes in its outcomes (Eiselt and Laporte 1989). Including such specifications into Hotelling's model may make its analytical investigation even more challenging and laborious. In this article we consider a discrete variant of the Hotelling game and study it by using Maple.

The remainder of this article is organised as follows: Section 2provides a (modest) literature overview of Hotelling's model. Section 3presents our discrete variant.Section 4 presents an analysis of a simple case and formulates some conjectures based on simulations with Maple. Conclusions are in Section 5.

2. Literature overview

Let us first consider the model as it was described by Hotelling in his seminal article on this topic (Hotelling 1929). The buyers of a certain commodity are distributed uniformly along a line with length *l*. On the line are two firms A and B, with respective distances *a* and *b* from the two extremes of the line. All buyers bring their bought commodities home at a cost of *c* per unit of distance. Hotelling makes some important further assumptions. He assumes that (1) the cost of production of both firms is zero; (2) one unit quantity of the commodity is sold in each unit of time, per unit of length; (3) no buyer has a preference for any of the sellers, all existing preferences are symbolized by the transportation price (and hence reflected in the 'final' price that is paid by the customer). Hotelling himself did not mention another important assumption he makes, namely that the transportation costs per unit product are taken to be independent of the number of units transported (Lerner and Singer 1937). The finding by Hotelling that the two firms will tend to locate as close together as possible at the centre of the line, became generally known as the Hotelling Law, or the principle of minimum differentiation (Eaton and Lipsey 1975). Hotelling's findings are later disputed in a stream of literature which started with Vickrey (1964) and attracted more attention after the formalisation of the arguments by d'Aspremont et al. (1979).

Stevens (1961) was the first to suggest that Hotelling's law can be treated as a game theoretical problem. Since then, most contributors to the literature on this law have combined Hotelling's main insights and propositions with later knowledge on equilibria in game theory. The easiest way to view Hotelling's model as a game, is to strictly divide his model into two parts, and make a game for each: a pricing game and a location game. In the Hotelling Pricing Game, firms can set prices and locations are fixed. In the Hotelling Location Game, firms can choose a location and prices are fixed (Rasmusen 2007).

Many amendments can be made to the original model of Hotelling. There is indeed a very wide body of literature in which adaptations of the model are examined, with a large variety of different outcomes. For a selective but useful overview, see Graitson (1982), Gabszewicz and Thisse (1992)or Kilkenny and Thisse (1999).

Apart from the assumptions Hotelling himself mentioned, there are a few assumptions that hold for almost all game-theoretical variations on the basic model: rational intelligent players, a symmetric game and complete information.

To obtain a general picture, authors that provide an overview of earlier literature usually categorize models according to the main assumptions made. Most important are the:

- number of firms;
- decision variables of these firms (location or price);
- order of the decisions (including whether or not the game is sequential);
- physical form of the market;
- economic form of the market (cooperation, oligopoly or pure competition);
- elasticity of demand;
- distribution of consumers;
- possibility for new entrants to enter the market.

The Hotelling game structure can also be relevant in cases where the locations do not represent geographical places, for example for models on elections for political parties; only the positioning is considered, prices are irrelevant (Hotelling 1929; Smithies 1941; Anderson et al. 1992; Negriu and Piatecki (2012)).In this context, a recent article is Ewerhart (2014).

The assumption of a uniform distribution of consumers is rather common in location models, probably because of its convenience for analysis (e.g. Tabuchi and Thisse 1995).

Eiselt (2011) points out a few directions for future research. He mentions the possibility of extensions to more dimensions, but immediately adds that it is doubtful that this is a promising niche, as the first results have shown it to be very difficult, even for pure locational models. Some have for example come up with circular or triangular markets to loosen the strict assumption of a linear market (Lerner and Singer, 1937; Vickrey, 1964; Eaton and Lipsey, 1975; Gupta et al, 2004). However, there are a few analyses with a two-dimensional space. There is a well-known conjecture of Eaton and Lipsey (1975) that for more than two firms in a two-dimensional space, an equilibrium with zero conjectural variation is unlikely to exist. However, Okabe and Suzuki (1987) show with Voronoi-diagrams that for very large numbers of firms, the configuration has stable equilibria in the inner area of square regions. Tabuchi (1994), Irmen and Thisse (1998) and Veendorp and Majeed (1995) all show that (under specific conditions) firms in a bounded two-dimensional space minimize differentiation on one dimension, and maximize it on the other.

We finish this (incomplete) literature overview by noting that discrete models, even in a one-dimensional space, are also scarce. An overview of the few existing studies can be found in Serra and Revelle (1994); also see Prisner (2011).

In the next section we will introduce a discrete Hotelling model in terms of a game in strategic form. As far as we know such a game has never been studied before.

3. A discrete Hotelling game

Below we present a discrete Hotelling game. We start by describing this game in one dimensional space.

3.1 One dimension

Consider the following game in strategic form with 2players, denoted by 1,2. The game depends on two parameters: a positive integer n and a real number w with $0 < w \leq 1$. Each player i has the same action set $X^i = \{0,1,\ldots,n\}$ considered as $n+1$ points on the real line and referred to as *locations*. The players simultaneously and independently choose a location; for $j=1,2$, player j chooses vertex x^j. Each of the vertices contributes as follows to the payoff the players. If a vertex is closest to one player, say at distance d, then this vertex contributes an amount w^d only to this player. If a vertex is closest to both players, say at distance d, then this vertex contributes $w^d/2$ to each player. We call w the *distance factor*.

It is straightforward to check that in case $w=1$ this leads to the following payoff functions $f^1, f^2 : X^1 \times X^2 \to \mathbb{R}$:

$$f^1(x^1, x^2) = \begin{cases} \dfrac{x^1 + x^2 + 1}{2} & \text{if } x^1 < x^2 \\ \dfrac{n+1}{2} & \text{if } x^1 = x^2 \\ n+1-\dfrac{x^1 + x^2 + 1}{2} & \text{if } x^1 > x^2, \end{cases}$$

$$f^2(x^1, x^2) = f^1(x^2, x^1).$$

So, for instance in case n = 10, the payoffs at some choices of vertices are:

$$f^1(0,10) = \frac{11}{2},\ f^2(0,10) = \frac{11}{2};\ f^1(4,5) = 5,\ f^2(4,5) = 6\ ;$$

$$f^1(3,3) = \frac{11}{2},\ f^2(3,3) = \frac{11}{2};\ f^1(3,8) = 6,\ f^2(3,8) = 5\ .$$

The reader is invited to check that for n = 4 and w = ½ the game is the following bi-matrix game with row (column) i + 1 for the action i of player 1 (2):

$$\begin{pmatrix} 31/32;31/32 & 1;15/8 & 5/4;2 & 3/2;2 & 13/8;13/8 \\ 15/8;1 & 19/16;19/16 & 3/2;7/4 & 7/4;7/4 & 2;3/2 \\ 2;5/4 & 7/4;3/2 & 5/4;5/4 & 7/4;3/2 & 2;5/4 \\ 2;3/2 & 7/4;7/4 & 3/2;7/4 & 19/16;19/16 & 15/18;1 \\ 13/8;13/8 & 3/2;2 & 5/4;2 & 1;15/8 & 31/32;31/32 \end{pmatrix}.$$

The corresponding set of Nash equilibria is $\{(1,2),(1,3),(2,3),(2,1),(3,1),(3,2)\}$.

3.2 More dimensions

Consider the game in Subsection 3.1. This game can be extended to two dimensions.

The only difference now is that the strategy spaces are squares with (n+1)(n+1) vertices; so each player has the strategy set $\{0,1,\ldots,n\} \times \{0,1,\ldots,n\}$.Distances are measured via the lattice (Manhattan distance). Of course, also generalisations to more than two dimensions are possible.

3.3 Interpretations

Of course, the above abstract interpretation allows for various more concrete economic interpretations. A possible one is the following.

Assume p^* and $q(p^*,t(d))$ represent constant price and quantity demanded respectively of the type of goods under consideration, with $t(d)$ for transportation costs per unit depending on the distance d between buyer and seller. In case:

$$q(p^*, t(d)) = \frac{w^{t(d)}}{p^*} \text{ and } t(d) = d,$$

we obtain for the contribution of a buyer to the pay-off of a seller at distance d:

$p^*q(p^*,t(d)) = w^d$

or an equal share of this if the location has an equal distance to the two sellers. So, in case there are no production costs, w^d is also the contribution to the profit of the seller.

4. Analysis and conjectures

4.1 One dimensional case and distance factor 1

Below we first present for the discrete Hotelling game in the case of one dimension and w=1 various fundamental observations in terms of three simple propositions and a theorem.

Proposition 1

1. The game is symmetric.
2. $f^1 + f^2 = n + 1$, thus thegame is a constant-sum game.

Proof

1. By definition $f^2(x^1,x^2) = f^1(x^2,x^1)$ for all (x^1,x^2). So the game is symmetric.
2. This immediately follows from the formulas for f^1 and f^2. Q.e.d.

As the game is a constant-sum game there are two consequences. (1) All strategy profiles are fully cooperative, i.e. maximise the total payoff function $f^1 + f^2$ (and therefore are Pareto efficient). (2) the game is strictly competitive, that is for all strategy profiles (a^1,a^2) and (b^1,b^2) it holds that $f^1(a^1,a^2) \leq f^1(b^1,b^2) \Leftrightarrow f^2(a^1,a^2) \geq f^2(b^1,b^2)$.

As the game is strictly competitive, it follows (see Friedman 1991) that:

Proposition 2

1. For all Nash equilibria (a^1,a^2) and (b^1,b^2) it holds that $f^1(a^1,a^2) = f^1(b^1,b^2)$ and $f^2(a^1,a^2) = f^2(b^1,b^2)$.
2. The set of Nash equilibria is a Cartesian product.

As the game is symmetric and the numbering of the vertices may be reversed, it follows that:

Proposition 3

If (e^1,e^2) is a Nash equilibrium, then also (e^2,e^1), $(n\text{-}e^1,n\text{-}e^2)$ and $(n\text{-}e^2,n\text{-}e^1)$ are Nash equilibria.

By now we do not know that the game has a Nash-equilibrium. However, it is straightforward to prove the following theorem:

*Theorem*1

In case w=1 the game has a symmetric Nash equilibrium. Moreover:

1. In case *n* is even, the set of Nash equilibria equals $\left\{\left(\frac{n}{2},\frac{n}{2}\right)\right\}$.
2. In case n is odd, the set of Nash equilibria equals
$$\left\{\left(\frac{n+1}{2},\frac{n-1}{2}\right),\left(\frac{n-1}{2},\frac{n+1}{2}\right),\left(\frac{n+1}{2},\frac{n+1}{2}\right),\left(\frac{n-1}{2},\frac{n-1}{2}\right)\right\}.$$

Proposition 2(1) guarantees that the payoff of player 1 is the same in each Nash-equilibrium; denote this value by *v*. The above makes it possible to determine *v* quickly: suppose (n^1,n^2) is a Nash-equilibrium. By Proposition 3,$(n^2,n^1)$is also a Nash equilibrium. By Proposition 1(1), $f^2(n^1,n^2)=f^1(n^2,n^1)$. By Proposition 2(1), $f^1(n^1,n^2)=f^1(n^2,n^1)$.Now $n+1=f^1(n^1,n^2)+f^2(n^1,n^2)=2f^1(n^1,n^2)$ and therefore $v=\frac{n+1}{2}$.

4.2 Conjectures

Based on our simulations with Maple we make the following conjectures for the one- and two-dimensional cases.

I. There exists for every n and w at least one Nash equilibrium.

II. In the one-dimensional case the average distance between the two players in the Nash equilibria decreases as *w* increases. The average distance is computed as the sum of all distances between the players in all possible Nash-equilibria divided by the number of these equilibria. Formally: fix n and denote by E(w) the set of Nash equilibria in case of distance factor w. Then
$$\frac{1}{\#E(w)}\cdot\sum_{(e^1,e^2)\in E(w)}\left|e^1-e^2\right| \quad \text{is a decreasing function of } w.$$

III. In the one-dimensional case, the two players aiming at maximum total payoff can do so by locating to about ¼ and ¾ of total space available. Formally, the strategy profile
$$\left(\left[\frac{n+1}{4}\right],\left[\frac{3(n+1)}{4}\right]\right)$$
is fully cooperative, i.e. maximises the total payoff.

5. Conclusion

It is not self-evident that discrete games have similar equilibria to their continuous counterparts. In this article we introduced a discrete version of the Hotelling game between two players with three parameters: the dimension (1 or 2), *n* (related to the number of vertices) and *w* (the distance factor). The one-dimensional case with distance factor 1 admits a straightforward game theoretical analysis. The analysis of the other cases is not at all straightforward. Based on simulation with the computer algebra program Maple we formulate some conjectures (for future research).

References

Anderson, S.P., De Palma, A. and Thisse, J-F. (1992), *Discrete Choice Theory of Product Differentiation*, MIT Press, Cambridge, Massachusetts.

d'Aspremont, C., Gabszewicz, J.J. and J.-F. Thisse, J-F. (1979), "On Hotelling's "Stability in competition"", *Econometrica*, Vol. 47, No. 5, pp. 1145-1150.

Eaton, B.C. and Lipsey, R.G. (1975), "The principle of minimum differentiation reconsidered: Some new developments in the theory of spatial competition", *The Review of Economic Studies*, Vol. 42, No. 1, pp. 27-49.

Ewerhart, C. (2014), Mixed equilibrium in a pure location game: the case of $n \geq 4$ firms, *Working paper No. 168*, University of Zurich.

Eiselt, H.A. (2011), "Equilibria in Competitive Location Models", in H.A. Eiselt and V. Marianov (eds.), *Foundations of Location Analysis*, Springer, New York, pp. pp. 139-162

Eiselt, H. A. and Laporte, G. (1989), "Competitive spatial models", *European Journal of Operational Research*, Vol. 39, pp. 231-242.

Friedman, J. (1986), *Game Theory with Applications to Economics*, Oxford University Press.

Gabszewicz, J.J. and Thisse, J.-F. (1992), "*Location*", in R. Aumann and S. Hart (eds.), *Handbook of Game Theory: with Economic Applications*, Vol. 1, Elsevier, Amsterdam, The Netherlands/ Oxford, UK, pp. 281-321

Graitson, D. (1982), Spatial competition à la Hotelling: aselective survey, *The Journal of Industrial Economics*, Vol. 31, No. 1/ 2, pp. 11-25.

Gupta, B., Lai, F-C., Pal, D., Sarkar, J. and Yu, C-M. (2004), "Where to locate in a circular city?", *International Journal of Industrial Organization*, Vol. 22, No. 6, pp. 759-782.

Hotelling, H. (1929), "Stability in competition", *The Economic Journal*, Vol. 39, No. 153, pp. 41-57.

Irmen, A. and Thisse, J-F. (1998), "Competition in multi-characteristics spaces: Hotelling was almost right", *Journal of Economic Theory*, Vol. 78, pp. 76-102.

Kilkenny, M. and Thisse, J-F. (1999), "Economics of location: a selective survey", *Computers and Operation Research*, Vol. 26, pp. 1369-1394.

Lerner, A.P. and Singer, H. W. (1937), "Some notes on duopoly and spatial competition", *The Journal of Political Economy*, Vol. 45, No. 2, pp. 145-186.

Negriu A. and Piatecki, C. (2012), "On the performance of voting systems in spatial voting simulations", *J Econ Interact Coord*, Vol. 7, pp. 63-77.

Okabe, A. and Suzuki, A. (1987), "Stability of spatial competition for a large number of firms on a bounded two-dimensional space", *Environment and Planning A*, Vol. 19, No. 8, pp. 1067-1082.

Prisner E. (2011), "Best response digraphs for two location games on graphs", in L.A. Petrosjan and N.A. Zenkevich (eds), *Contributions to Game Theory and Management*, Vol. 4, St. Petersburg, pp. 378-388.

Rasmusen, E. (2007), *Games and information; an introduction to game theory*, 4th edition, Blackwell, Malden, Massachusetts.

Serra, D. and Charles ReVelle, C., (1994), "Competitive location in discrete space", *Economics Working Paper Series*, 96, in Z. Drezner (ed.), *Facility Location: A Survey of Applications and Methods*, Springer-Verlag, New York, pp. 367-386.

Smithies, A. (1941), "Optimum location in spatial competition", *The Journal of Political Economy*, Vol. 49, No. 3, pp. 423-439.

Stevens, B.H. (1961), "An application of game theory to a problem in location strategy", *Papers and Proceedings of the Regional Science Association*, Vol. 7, pp. 143-157.

Tabuchi, T. (1994), "Two-stage, two-dimensional spatial competition between two firms", *Regional Science and Urban Economics*, Vol. 24, pp. 207-227.

Tabuchi, T. and Thisse, J-F. (1995), "Asymmetric equilibria in spatial competition", *International Journal of Industrial Organization*, Vol. 13, pp. 213-227.

Veendorp, E.C.H. and Majeed, A. (1995), "Differentiation in a two-dimensional market", *Regional Science and Urban Economics*, Vol. 25, pp. 75-83.

Vickrey, W. (1964), *Microstatics*, Harcourt, Brace and World, New York.